Elliptically Contoured Models in Statistics and Portfolio Theory

Arjun K. Gupta • Tamas Varga • Taras Bodnar

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Dedicated to the memory of my parents the late Smt. Leela and Sh. Amarnath Gupta.

(AKG)

Dedicated to my children Terézia, Julianna, and Kristóf.

(TV)

Dedicated to my wife Olha and to my children Bohdan and Anna-Yaroslava.

(TB)

Preface

In multivariate statistical analysis, elliptical distributions have recently provided an alternative to the normal model. Most of the work, however, is spread out in journals throughout the world and is not easily accessible to the investigators. Fang, Kotz, and Ng presented a systematic study of multivariate elliptical distributions; however, they did not discuss the matrix variate case. Fang and Zhang have summarized the results of generalized multivariate analysis which include vector as well as the matrix variate distributions. On the other hand, Fang and Anderson collected research papers on matrix variate elliptical distributions, many of them published for the first time in English. They published very rich material on the topic, but the results are given in paper form which does not provide a unified treatment of the theory. Therefore, it seemed appropriate to collect the most important results on the theory of matrix variate elliptically contoured distributions available in the literature and organize them in a unified manner that can serve as an introduction to the subject.

The book will be useful for researchers, teachers, and graduate students in statistics and related fields whose interests involve multivariate statistical analysis and its application into portfolio theory. Parts of this book were presented by Arjun K. Gupta as a one semester course at Bowling Green State University. Knowledge of matrix algebra and statistics at the level of Anderson is assumed. However, Chap. 1 summarizes some results of matrix algebra. This chapter also contains a brief review of the literature and a list of mathematical symbols used in the book.

Chapter 2 gives the basic properties of the matrix variate elliptically contoured distributions, such as the probability density function and expected values. It also presents one of the most important tools of the theory of elliptical distributions, the stochastic representation.

The probability density function and expected values are investigated in detail in Chap. 3.

Chapter 4 focuses on elliptically contoured distributions that can be represented as mixtures of normal distributions.

The distributions of functions of random matrices with elliptically contoured distributions are discussed in Chap. 5. Special attention is given to quadratic forms.

Characterization results are given in Chap. 6.

The next three chapters are devoted to statistical inference. Chapter 7 focuses on estimation results, whereas Chap. 8 is concerned with hypothesis testing problems. Inference for linear models is studied in Chap. 9.

Chapter 10 deals with the application of the elliptically contoured distributions for modeling financial data. We present distributional properties of the estimated main characteristics of optimal portfolios, like variance and expected return assuming that the asset returns are elliptically contoured distributed. The joint distributions of the estimated parameters of the efficient frontier are derived as well as we provide exact inference procedures for the corresponding population values. We also study the distributional properties of the estimated weights of the global minimum variance portfolio in detail.

In Chap. 11, we consider a further extension of matrix variate elliptically contoured distributions that allows us to model the asymmetry in data. Here, first the multivariate skew normal distribution is presented and its matrix generalization is discussed. We also study the main properties of this distribution, like moments, the density function, and the moment-generating function. Next, the skew t-distribution is introduced as well as the general class of matrix variate skew elliptically contoured distributions. Moreover, we present the distributional properties of quadratic forms in skew elliptical distributions and discuss the inference procedures. An application into portfolio theory is discussed as well. Finally, an up-to-date bibliography has been provided, along with author and subject indexes. The materials in the first nine chapters are from the book Elliptically Contoured Models in Statistics by the first two authors. The material in Chaps. 10 and 11 is taken from the papers of the authors. Permission of their publishers Kluwer Academic Publishers (http://www.wkap.com), Japan Statistical Society (http://www.jss.gr.jp), Springer (http://www.springer.com), and Taylor and Francis (http://www.tandfonline.com/) is gratefully acknowledged.

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Acronyms

We denote matrices by capital bold letters, vectors by small bold letters and scalars by small letters. We use the same notation for a random variable and its values. Also the following notations will be used in the sequel.

$\mathbb{R}^p$	the p-dimensional real space
$\mathscr{B}(\mathbb{R}^p)$	the Borel sets in $\mathbb{R}^p$
S_p	the unit sphere in $\mathbb{R}^p$
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}_0^+$	the set of nonnegative real numbers
$\chi_A(x)$	the indicator function of A, that is $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$
$\chi(x \geq t)$	the same as $\chi_{[t,\infty)}(x)$ (t is a real number)
$\mathbf{A} \in \mathbb{R}^{p \times n}$	$\mathbf{A}$ is a $p \times n$ real matrix
a_{ij}	the (i, j)th element of matrix $\mathbf{A}$
$\mathbf{A}'$	transpose of $\mathbf{A}$
$rk(\mathbf{A})$	rank of $\mathbf{A}$
$\mathbf{A} > \mathbf{0}$	the square matrix $\mathbf{A}$ is positive definite (see also Sect. 1.2)
$\mathbf{A} \geq \mathbf{0}$	the square matrix $\mathbf{A}$ is positive semidefinite (see also Sect. 1.2)
$\|\mathbf{A}\|$	determinant of the square matrix $\mathbf{A}$
$tr(\mathbf{A})$	trace of the square matrix $\mathbf{A}$
$etr(\mathbf{A})$	$\exp(tr(\mathbf{A}))$ if $\mathbf{A}$ is a square matrix
$\\|\mathbf{A}\\|$	norm of $\mathbf{A}$ defined by $\\|\mathbf{A}\\| = \sqrt{tr(\mathbf{A}'\mathbf{A})}$
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$
$\mathbf{A}^-$	generalized inverse of $\mathbf{A}$, that is $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$ (see also Sect. 1.2)
$\mathbf{A}^{1/2}$	let the spectral decomposition of $\mathbf{A} \geq 0$ be $\mathbf{GDG}'$, and define $\mathbf{A}^{1/2} = \mathbf{GD}^{1/2}\mathbf{G}'$ (see also Sect. 1.2)
$O(p)$	the set of $p \times p$ dimensional orthogonal matrices
$\mathbf{I}_p$	the $p \times p$ dimensional identity matrix
$\mathbf{e}_p$	the p-dimensional vector whose elements are 1's; that is, $\mathbf{e}_p = (1, 1, \ldots, 1)'$ real matrix
$\mathbf{A} \otimes \mathbf{B}$	Kronecker product of the matrices $\mathbf{A}$ and $\mathbf{B}$ (see also Sect. 1.2)

$\mathbf{A} > \mathbf{B}$	$\mathbf{A} - \mathbf{B}$ is positive definite
$\mathbf{A} \geq \mathbf{B}$	$\mathbf{A} - \mathbf{B}$ is positive semidefinite
$vec(\mathbf{A})$	the vector $\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$ where $\mathbf{a}_i$ denotes the ith column of $p \times n$ matrix $\mathbf{A}$, $i = 1, 2, \ldots, n$
$J(\mathbf{X} \rightarrow f(\mathbf{X}))$	the Jacobian of the matrix transformation f
$\mathbf{X} \sim \mathscr{D}$	the random matrix $\mathbf{X}$ is distributed according to the distribution $\mathscr{D}$
$\mathbf{X} \approx \mathbf{Y}$	the random matrices $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed
$Cov(\mathbf{X})$	covarinace matrix of the random matrix $\mathbf{X}$; that is $Cov(\mathbf{X}) = Cov(vec(\mathbf{X}'))$
$\phi_{\mathbf{X}}(\mathbf{T})$	the characteristic function of the random matrix $\mathbf{X}$ at $\mathbf{T}$; that is $E(etr(i\mathbf{T}'\mathbf{X}))$, $\mathbf{X}, \mathbf{T} \in I\!R^{p \times n}$

For a review of Jacobians, see Press (1972) and Siotani, Hayakawa and Fujikoshi (1985). We also use the following notations for some well known probability distributions.

UNIVARIATE DISTRIBUTIONS:

$N(\mu, \sigma^2)$ normal distribution; its probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

where $\mu \in I\!R$, $\sigma \in I\!R^+$, and $x \in I\!R$

$B(a,b)$ beta distribution; its probability density function is

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1},$$

where $a > 0$, $b > 0$, $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and $0 < x < 1$

t_n Student's t-distribution; its probability density function is

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}},$$

where $n > 0$, and $x \in I\!R$

χ^2_n chi-square distribution; its probability density function is

$$f(x) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} \exp\left\{-\frac{x}{2}\right\},$$

where $n > 0$, and $x \geq 0$

χ_n chi distribution; its probability density function is

$$f(x) = \frac{1}{2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)} x^{n-1} \exp\left\{-\frac{x^2}{2}\right\},$$

where $n > 0$, and $x \geq 0$

$F_{n,m}$ F distribution; its probability density function is

$$f(x) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{n}{m}\right)^{\frac{n}{2}} \frac{x^{\frac{n}{2}-1}}{\left(1+\frac{n}{m}x\right)^{\frac{n+m}{2}}},$$

where $n, m = 1, 2, \ldots,$, and $x > 0$

$U_{p,m,n}$ U distribution, which is the same as the distribution of $\prod_{i=1}^{p} v_i$; where v_i's are independent and

$$v_i \sim B\left(\frac{n+1-i}{2}, \frac{m}{2}\right)$$

For the U distribution, see Anderson (2003), pp. 307–314.

MULTIVARIATE DISTRIBUTIONS:

$N_p(\mu, \Sigma)$ multivariate normal distribution; its characteristic function is

$$\phi_{\mathbf{x}}(\mathbf{t}) = \exp\left\{i\mathbf{t}'\mu + \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\right\},$$

where $\mathbf{x}, \mathbf{t}, \mu \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p \times p}$, and $\Sigma \geq \mathbf{0}$

$D(m_1, \ldots, m_p; m_{p+1})$ Dirichlet distribution; its probability density function is

$$f(\mathbf{x}) = \frac{\Gamma\left(\sum_{i=1}^{p+1} m_i\right)}{\prod_{i=1}^{p+1} \Gamma(m_i)} \prod_{i=1}^{p} x_i^{m_i - 1} \left(1 - \sum_{i=1}^{p} x_i\right)^{m_{p+1}-1},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_p)' \in \mathbb{R}^p$, $0 < \sum_{i=1}^{p} x_i < 1$, and $m_i > 0$, $i = 1, 2, \ldots, p$

$SMT_v(\alpha)$ multivariate skew t-distribution; its probability density function is

$$f_v(\mathbf{y}) = 2 f_{T_v}(\mathbf{y}) F_{T_{v+p}} \left(\frac{\alpha' \mathbf{y}}{(v + \mathbf{y}' \Sigma^{-1} \mathbf{y})^{\frac{1}{2}}} \sqrt{v+p} \right),$$

where $f_{T_k}(\cdot)$ and $F_{T_k}(\cdot)$ denote the probability density function and the cumulative distribution function of central t-distribution with k degrees of freedom, respectively; $\mathbf{y} \in \mathbb{R}^p$, $v > 0$, $\alpha \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p \times p}$, and $\Sigma > \mathbf{0}$.

$SMC(\alpha)$ multivariate skew Cauchy distribution; its probability density function is

$$f(\mathbf{y}) = \frac{2\Gamma\left(\frac{p+1}{2}\right)}{(\pi)^{\frac{p+1}{2}}} \left(1 + \sum_{1}^{p} y_j^2\right)^{-\frac{p+1}{2}} F_{T_{p+1}} \left(\frac{\alpha' \mathbf{y} \sqrt{p+1}}{\left(1 + \sum_{1}^{p} y_j^2\right)^{\frac{1}{2}}} \right),$$

where $F_{T_k}(\cdot)$ denotes the cumulative distribution function of central t-distribution with k degrees of freedom; $\mathbf{y} \in \mathbb{R}^p$, $\alpha \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p \times p}$, and $\Sigma > \mathbf{0}$.

$CSN_{p,q}(\mu, \Sigma, \mathbf{D}, \nu, \Delta)$ closed skew normal distribution; its probability density function is

$$g_{p,q}(\mathbf{y}) = C \phi_p(\mathbf{y}; \mu, \Sigma) \Phi_q[\mathbf{D}(\mathbf{y} - \mu); \nu, \Delta],$$

with

$$C^{-1} = \Phi_q[\mathbf{0}; \nu, \Delta + \mathbf{D}\Sigma\mathbf{D}'] \tag{1}$$

where $\phi_l(\mathbf{x}; \mu, \Sigma)$ and $\Phi_l(\mathbf{x}; \mu, \Sigma)$ denote the probability density function and the cumulative distribution function of the l-dimensional normal distribution with mean vector μ and covariance matrix Σ, respectively; $\mathbf{y} \in \mathbb{R}^p$, $p, q \geq 1$, $\mu \in \mathbb{R}^p$, $\nu \in \mathbb{R}^q$, $\mathbf{D} \in \mathbb{R}^{q \times p}$, $\Sigma \in \mathbb{R}^{p \times p}$, $\Sigma > \mathbf{0}$, $\Delta \in \mathbb{R}^{q \times q}$, and $\Delta > \mathbf{0}$.

MATRIX VARIATE DISTRIBUTIONS:

$N_{p,n}(\mathbf{M}, \Sigma \otimes \Phi)$ matrix variate normal distribution; its characteristic function is

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr\left\{i\mathbf{T}'\mathbf{M} + \frac{1}{2}\mathbf{T}'\Sigma\mathbf{T}\Phi\right\},$$

where $\mathbf{M}, \mathbf{X}, \mathbf{T} \in I\!R^{p\times n}$, $\Sigma \in I\!R^{p\times p}$, $\Sigma \geq \mathbf{0}$, $\Phi \in I\!R^{n\times n}$, and $\Phi \geq \mathbf{0}$

$W_p(\Sigma, n)$ Wishart distribution; its probability density function is

$$f(\mathbf{X}) = \frac{|\mathbf{X}|^{\frac{n-p-1}{2}} etr\left\{-\frac{1}{2}\Sigma^{-1}\mathbf{X}\right\}}{2^{\frac{np}{2}}|\Sigma|^{\frac{n}{2}}\Gamma_p\left(\frac{n}{2}\right)},$$

where $\mathbf{X} \in I\!R^{p\times p}$, $\mathbf{X} > \mathbf{0}$, $\Sigma \in I\!R^{p\times p}$, $\Sigma > \mathbf{0}$, p, n are integers, $n \geq p$, and

$$\Gamma_p(t) = \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^{p} \Gamma\left(t - \frac{i-1}{2}\right)$$

$B_p^I(a,b)$ matrix variate beta distribution of type I; its probability density function is

$$f(\mathbf{X}) = \frac{|\mathbf{X}|^{a-\frac{p+1}{2}}|\mathbf{I}_p - \mathbf{X}|^{b-\frac{p+1}{2}}}{\beta_p(a,b)},$$

where $a > \frac{p-1}{2}$, $b > \frac{p-1}{2}$, $\beta_p(a,b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}$, $\mathbf{X} \in I\!R^{p\times p}$, and $\mathbf{0} < \mathbf{X} < \mathbf{I}_p$

$B_p^{II}(a,b)$ matrix variate beta distribution of type II; its probability density function is

$$f(\mathbf{X}) = \frac{|\mathbf{X}|^{a-\frac{p+1}{2}}|\mathbf{I}_p + \mathbf{X}|^{-(a+b)}}{\beta_p(a,b)},$$

where $a > \frac{p-1}{2}$, $b > \frac{p-1}{2}$, $\mathbf{X} \in I\!R^{p\times p}$, and $\mathbf{X} > \mathbf{0}$

$T_{p,n}(m, \mathbf{M}, \Sigma, \Phi)$ matrix variate T distribution; its probability density function is

$$f(\mathbf{X}) = \frac{\pi^{-\frac{np}{2}}\Gamma_p\left(\frac{n+m+p-1}{2}\right)}{\Gamma_p\left(\frac{m+p-1}{2}\right)|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} |\mathbf{I}_p + \Phi^{-1}(\mathbf{X}-\mathbf{M})\Sigma^{-1}(\mathbf{X}-\mathbf{M})'|^{-\frac{n+m+p-1}{2}},$$

where $m > 0$, $\mathbf{M}, \mathbf{X}, \mathbf{T} \in I\!R^{p\times n}$, $\Sigma \in I\!R^{p\times p}$, $\Sigma > \mathbf{0}$, $\Phi \in I\!R^{n\times n}$, and $\Phi > \mathbf{0}$

$E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ matrix variate elliptically contoured distribution; its characteristic function

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr(i\mathbf{T}'\mathbf{M})\,\psi(tr(\mathbf{T}'\Sigma\mathbf{T}\Phi))\,,$$

where $\mathbf{T} : p \times n$, $\mathbf{M} : p \times n$, $\Sigma : p \times p$, $\Phi : n \times n$, $\Sigma \geq \mathbf{0}$, $\Phi \geq \mathbf{0}$, and $\psi : [0, \infty) \to \mathbb{R}$

For further discussion of $B_p^I(a,b)$, $B_p^{II}(a,b)$, see Olkin and Rubin (1964) and Javier and Gupta (1985b), and for results on $T_{p,n}(m, \mathbf{M}, \Sigma, \Phi)$, see Dickey (1967).